Dynamics of the Quantum Duffing Oscillator

Matthew G. Mansell, Alma College; Lawrenzo Moses, The University of Akron; and Dr. Bradley Trees, Ohio Wesleyan University

The Problem

We studied a periodically driven, damped, weakly nonlinear, quantum oscillator and compared to its classical analog in an attempt to model and further understand the dynamics of Josephson junctions.

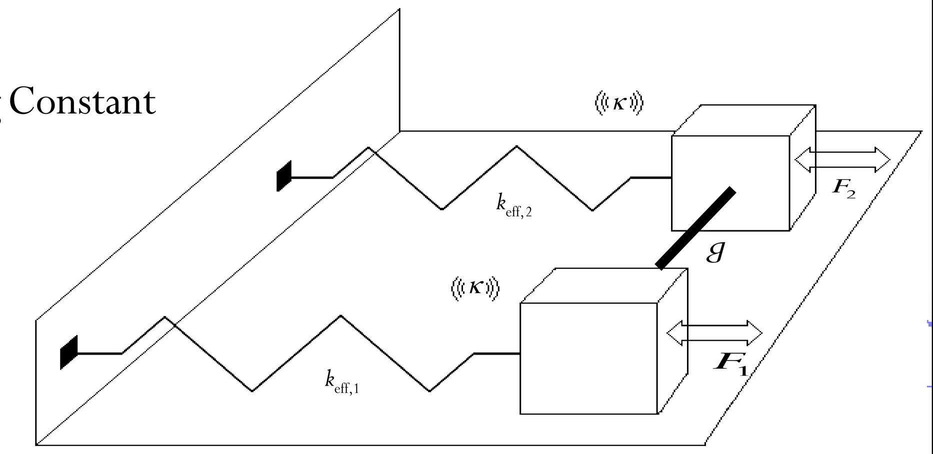
$k_{eff} = k + \varepsilon x^2$ = Effective Spring Constant

F_i = Periodic Driving Force

κ = Damping Coefficient

g = Coupling Coefficient

m_i = Mass i



Classical Synchronization

Newton's equations of motion for periodically driven, damped, nonlinear, coupled oscillators:

$$\frac{d^2x_1}{dt^2} + \kappa x_1 + \omega_\circ^2 x_1 + \varepsilon x_1^3 - g(x_1 - x_2) = F_1\cos(\omega_d t)$$

$$\frac{d^2x_2}{dt^2} + \kappa x_2 + \omega_\circ^2 x_2 + \varepsilon x_2^3 - g(x_1 - x_2) = F_2\cos(\omega_d t)$$

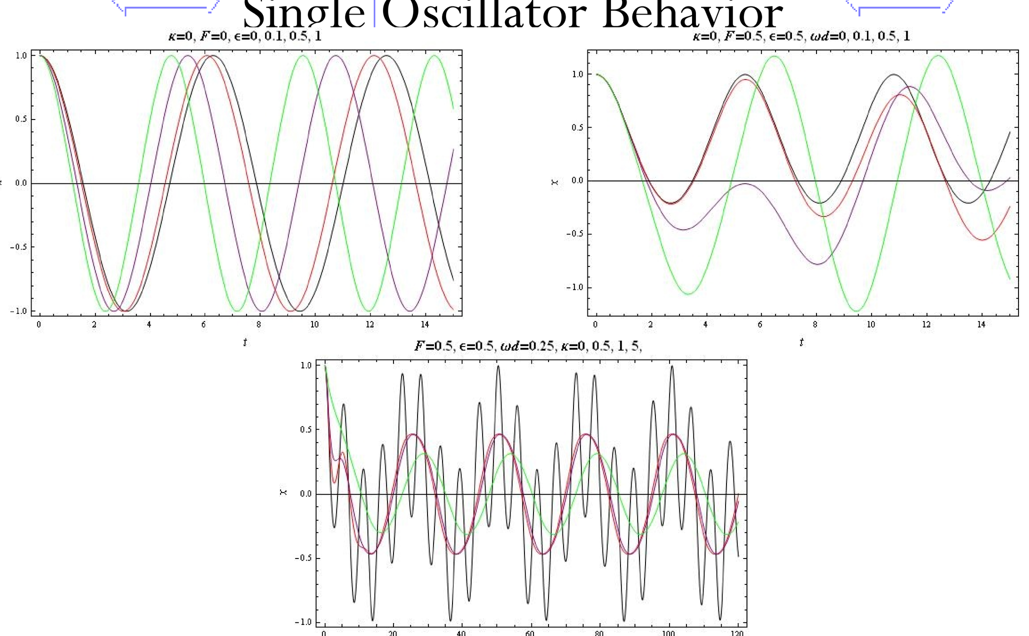
κ = Damping Parameter, ε = Nonlinear Parameter, g = Coupling Strength

x_1 = Position of Mass 1, F_1 = Driving Force Amplitude on Mass 1

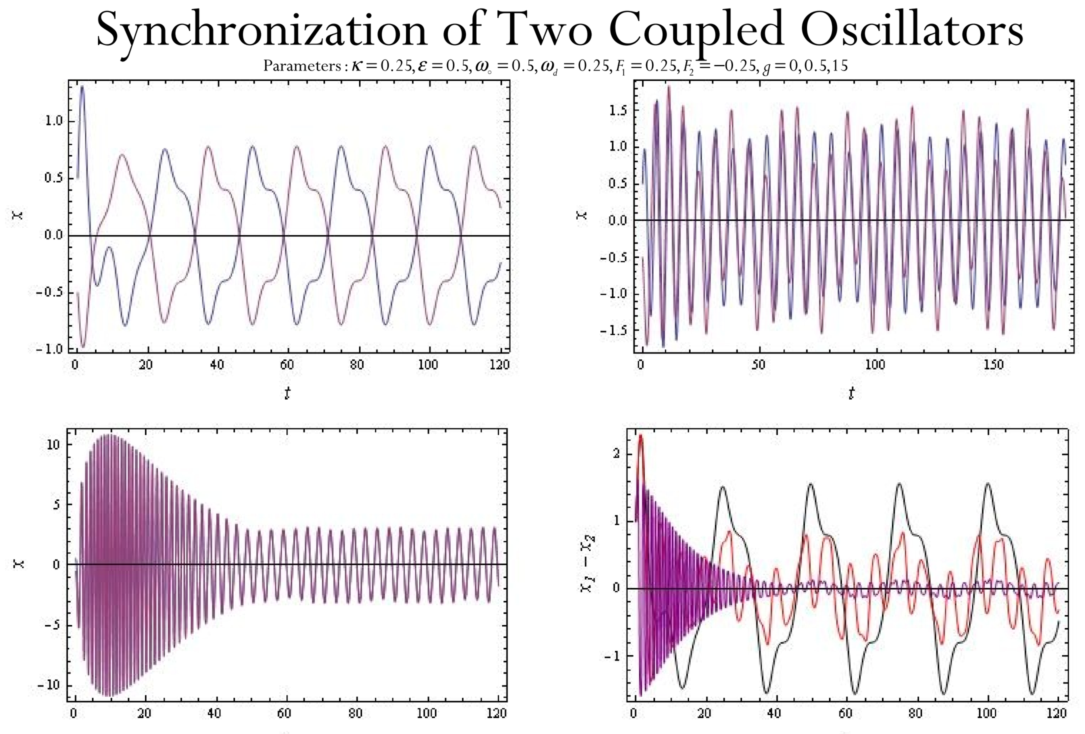
x_2 = Position of Mass 2, F_2 = Driving Force Amplitude on Mass 2

$\omega_\circ$ = Natural Oscillator Frequency, ω_d = Driving Frequency

Single Oscillator Behavior



Synchronization of Two Coupled Oscillators



Methodology: Quantum Case

- Solve Schrödinger's equation for a periodically driven, nonlinear, quantum oscillator using a Hamiltonian operator of the form:

$$\hat H = \underbrace{\frac{p^2}{2m}}_{\text{Kinetic Energy}} + \underbrace{\tfrac{1}{2}kx^2}_{\text{Harmonic Potential Energy}} + \underbrace{\varepsilon x^4}_{\text{Non-linearity}} + \underbrace{F\cos(\omega_d t)}_{\text{Periodic Driving Force}}.$$

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \hat H\Psi(x,t),$$

- Damping is added to Schrödinger's equation using the Quantum State Diffusion (QSD) method.
  - This method involves a system (the oscillator) surrounded by an environment. The environment is much larger than the system, such that the environment can affect the system; however, the system cannot affect the environment. The equation is constructed by considering the average effect of the environment on the system. After inserting the QSD damping term we obtain a modified Schrödinger's Equation:

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \hat H\Psi(x,t) + \left(\langle\hat L^+\rangle_\psi\hat L - \frac{1}{2}\hat L^+\hat L - \frac{1}{2}\langle\hat L^+\rangle_\psi\langle\hat L\rangle_\psi\right)\Psi(x,t) + \left(\hat L - \langle\hat L\rangle_\psi\right)\Psi(x,t)\frac{d\xi}{dt}.$$

  - The $\hat L$ and $\hat L^+$ are called Lindblad operators, which are linear combinations of the position and momentum operators.

$$\hat L = \sqrt{\kappa}\,a = \sqrt{\kappa}\sqrt{\frac{1}{2\hbar m\omega_0}}\left(ip + m\omega_0 x\right)$$

$$\hat L^+ = \sqrt{\kappa}\,a^+ = \sqrt{\kappa}\sqrt{\frac{1}{2\hbar m\omega_0}}\left(-ip + m\omega_0 x\right).$$

where a and a^+ are the ladder operators for the simple harmonic oscillator.

  - The $\partial\xi$ in the last term is a complex differential Gaussian random variable centered at zero. This number is used to determine the extent to which the system interacts with the environment.

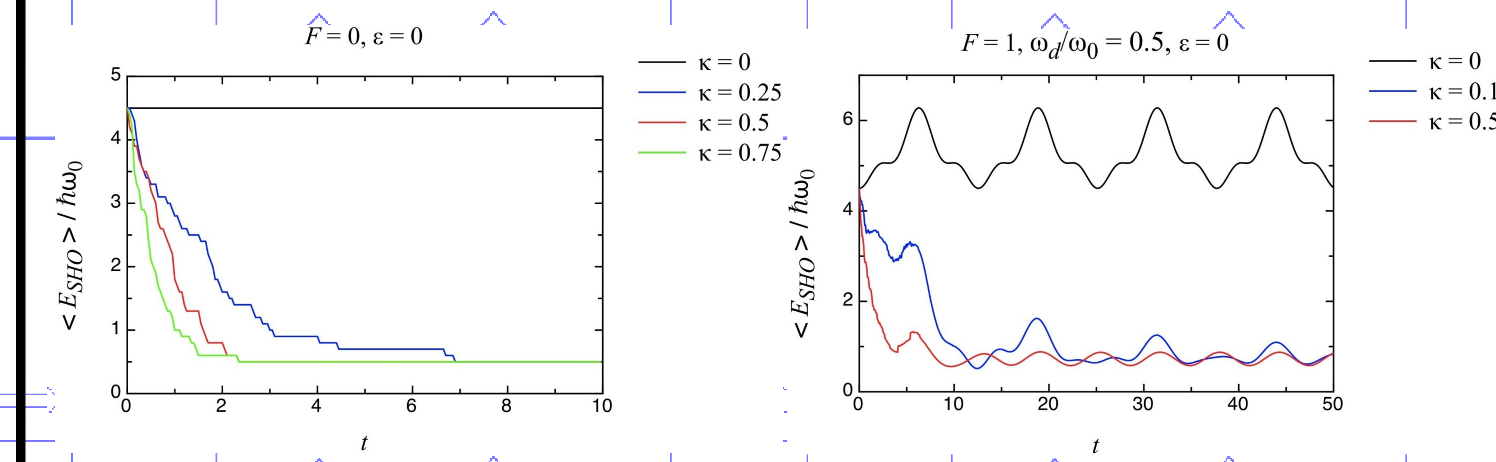
- Calculate various expectation values.
  - Choose $\Psi(x,0) = \psi_n(x)$, which are energy eigenstates of the Simple Harmonic Oscillator.
  - Solve Schrödinger's Equation numerically for $\Psi(x,t)$ using the fourth-order Runge-Kutta method, after which we calculate expectation values such as:

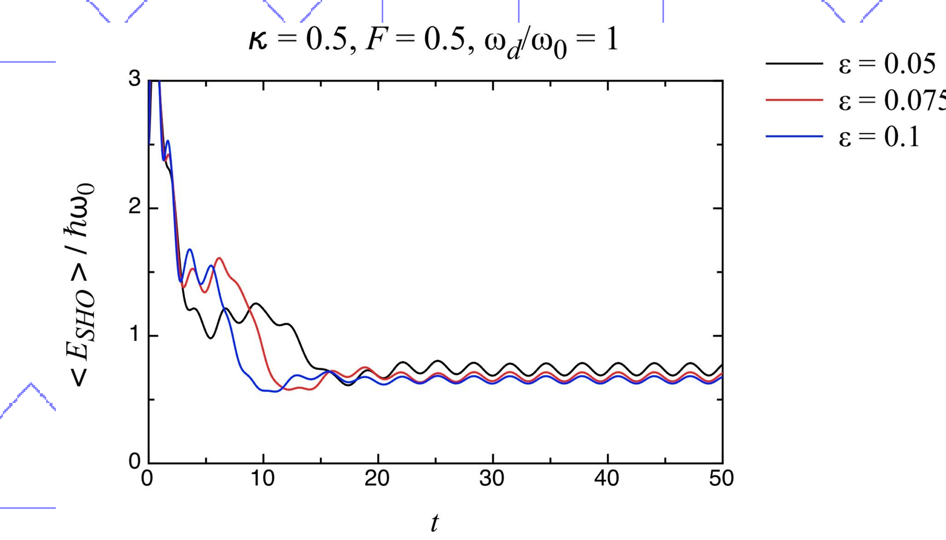
$$\langle x\rangle(t) = \int_{-\infty}^{\infty}\Psi^*(x,t)\,\hat x\,\Psi(x,t)\,dx,\quad \langle x^2\rangle(t) = \int_{-\infty}^{\infty}\Psi^*(x,t)\,\hat x^2\,\Psi(x,t)\,dx$$

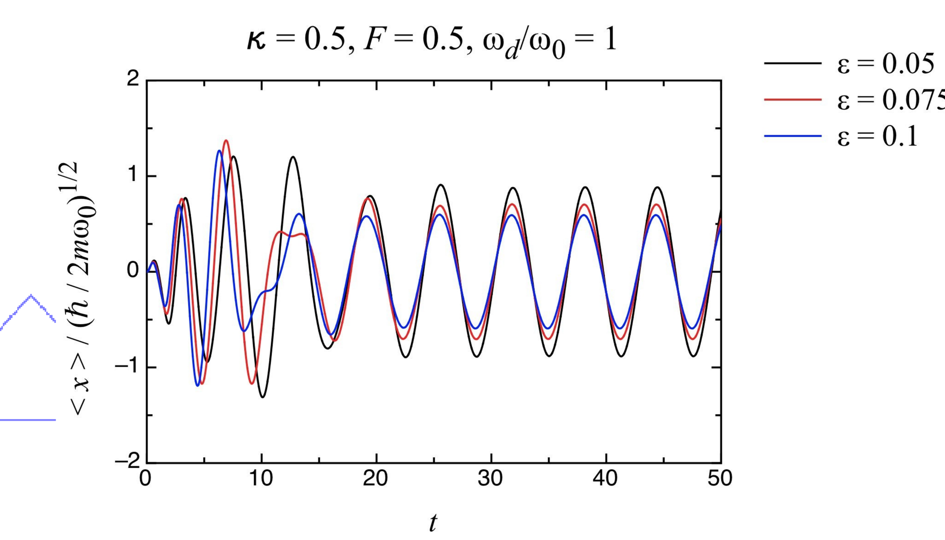
$$\langle p\rangle(t) = \int_{-\infty}^{\infty}\Psi^*(x,t)\,\hat p\,\Psi(x,t)\,dx,\quad \langle p^2\rangle(t) = \int_{-\infty}^{\infty}\Psi^*(x,t)\,\hat p^2\,\Psi(x,t)\,dx$$

$$\langle E_{Mech}\rangle \approx \langle E_{SHO}\rangle = \frac{\langle p^2\rangle}{2m} + \tfrac{1}{2}m\omega_0^2\langle x^2\rangle$$

Results: Quantum Duffing Oscillator







Future Plans

- Plans:
  - Solve Schrödinger's equation for two coupled, quantum Duffing oscillators.
  - Look for evidence of quantum synchronization.
  - Comments:
  - Fourth-order Runge-Kutta may not adequately handle coupled nonlinear quantum oscillators.
  - The nonlinear oscillators are numerically unstable.

Acknowledgements

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