Shared Shortest Paths in Graphs

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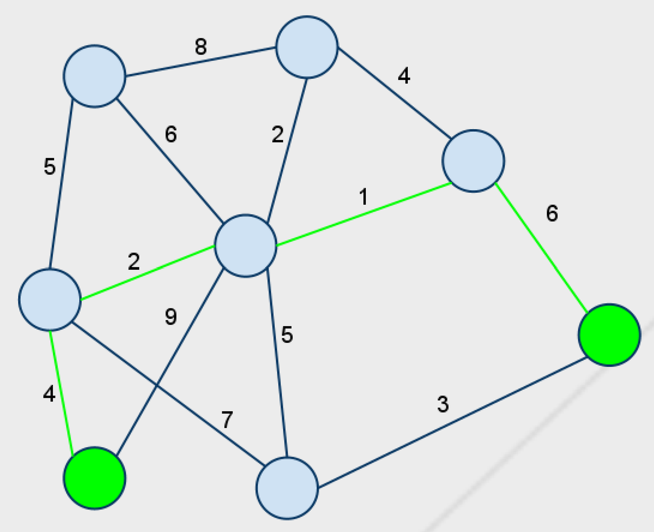
Graph Theory

Basics

A graph is a set of vertices and a set of edges. Each edge consists of a pair of verticies.

In the case of the SSPP, graphs are weighted meaning in addition to a pair of verticies each edge has a cost associated to it.

Graphs may be directed or undirected, meaning that travel is not necessarily allowed forward and backward.



A weighted, undirected graph with a path highlighted in green.

Paths and Journeys

A series of connected vertices forms a path.

The cost of a path is determined by summing the weights of the edges between verticies on the path.

A shortest path between two vertices is the shortest possible path between the two vertices.

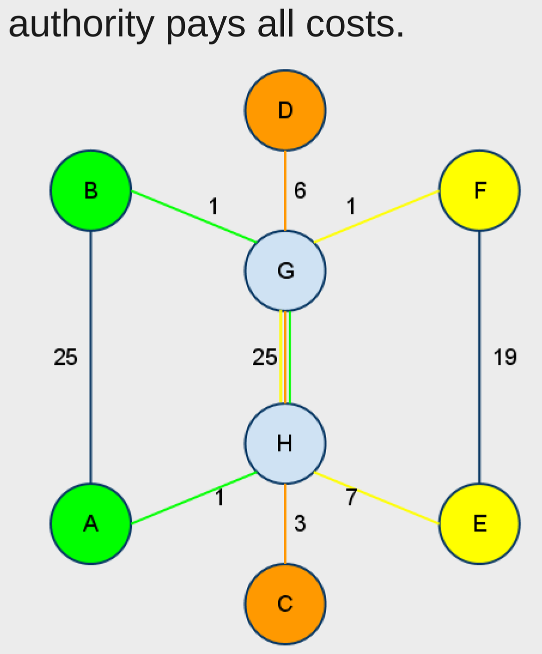
In the SSPP a Journey is attempting to travel between two vertices, usually along the shortest possible path.

Shared Shortest Path Problem

The Shared Shortest Path Problem asks what is the best way, considering that journeys that use the same edge split the cost of the edge evenly, to route journeys based on some metric.

One measurement of success is whether or not the current configuration is a (Strong) Nash equilibrium. These configurations are noted as good because they are stable in that agents are unlikely to change their paths.

Another way to approach the problem is to attempt to minimize the total cost for all journeys. These situations are beneficial in cases where a central authority pays all costs.



Costs:

Total Cost : 44
Strong Nash Equilibrium

Journey 1 :
1 + 25/3 + 1 = 31/3

Journey 2 :
3 + 25/3 + 6 = 52/3

Journey 3 :
7 + 25/3 + 1 = 49/3

Background

The Shared Shortest Path Problem (SSPP) asks how to route paths in a graph to minimize cost when paths split evenly the costs of journeys they mutually use.

Game Theory

One method of analysis takes a game theoretic approach to the problem, treating each individual journey as a selfish agent. This approach has the benefit of modeling real world situations without a central authority more accurately than a total cost analysis would.

Some concepts from Game Theory include:

Nash Equilibrium

A Nash equilibrium is a configuration of choices by players such that no individual player (journey) can improve their cost by taking a different path.

Strong Nash Equilibrium

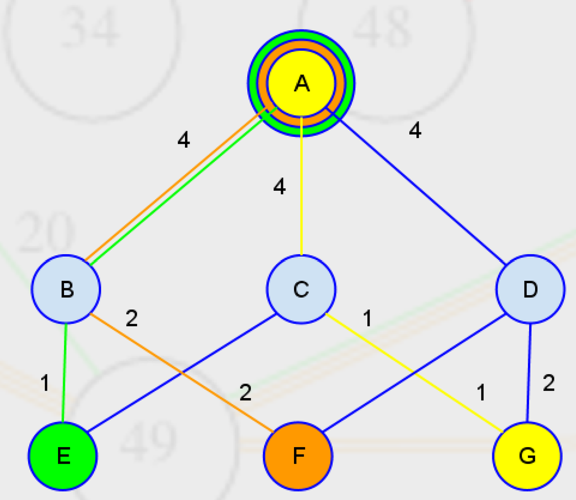
A Strong Nash Equilibrium is a configuration in which no group of journeys can change their positions to improve the costs of all members of the group.

Existence of Equilibria

The SSPP can be expressed as a special type of game known as a Congestion Game.

Among other things, this property ensures that at least one Nash Equilibrium always exists in any instance of the game.

While Nash Equilbria are guaranteed to exist in all graphs, there are known cases in which Strong Nash Equilibria do not exist.



Price of Stability

Because a Nash equilibrium is a "stable" state in the sense that individual agents will have no desire to affect it is a natural question to ask how close in total cost a configuration that is a Nash Equilibrium can get in total cost to the minimum solution.

In the case of directed graphs, it is known that this “price of stability” grows logarithmicaly in the number of players on the graph.

For undirected graphs it is an open question whether or not the previous bound holds.

NP-Complete

Finding minimum total cost solutions to the SSPP is classified as an NP-Complete problem.

This can be determined by a simple polynomial time reduction from the Steiner Tree Problem which asks what is the minimum total weight tree that connects a set of points R. These points are then treated as start and destination points of journeys in an instance of the SSPP problem.

Because finding exact minimum solutions is NP-Complete it is likely intractable to do and other methods must be employed.

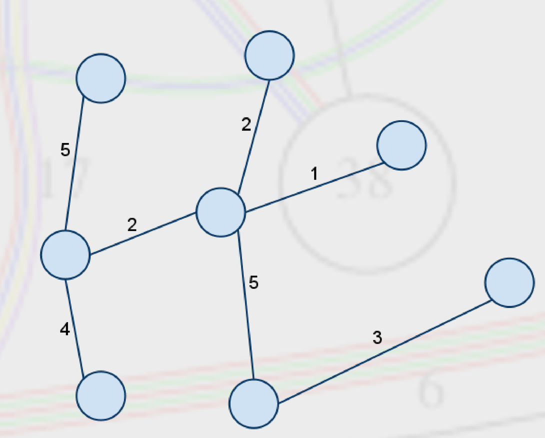
Heuristics

Although finding exact minimum total cost solutions is likely to be an intractable problem it is possible to heuristically find approximate solutions whose total cost is close to an optimal solution.

The following approximation methods are employed to find near optimal solutions:

Spanning Tree Heuristic

This approximation method considers only edges on a minimum spanning tree of the original graph. All journeys are then routed on the only paths available to them.



The minimum spanning tree of the graph in Column 2.

If Graphs that satisfy a reasonable property, the triangle inequality, this heuristic is guaranteed to find solutions at worst 2 times the cost of the optimum solution.

DEASE Algorithm

The DEASE algorithm (short for Delete Edge And Share Edge) attempts to encourage sharing between groups that may not share based on the local landscape of greedy options by making unsuable edges shared by a certain number of journeys.

After an edge is deleted journeys make new path choices based on the current state of the graph.

It is hoped that these edge deletions will lead to a Strong Nash Equilibrium, if it exists.

Nash Equilibrium Finder

A Nash Equilibrium can be found algorithmically from any valid configuration, such as one derived from the above approximation methods, of the SSPP game.

The algorithm for finding an equilibrium simply checks whether any journey can make a better path choice and moves this journey if it can.

Because the SSPP game is a congestion game this action repeatedly decreases what is known as the potential of the solution. Once the potential has reach a local minimum, the current state is a Nash equilibrium.

With this reasoning the previous algorithm is guaranteed to halt in a finite number of deviations.

The growth of the number of deviations remains an open question.

Applications

The SSPP has applications in network design, and group network routing.

The positioning of stops in a city metro is one application of the SSPP.

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